NEUTRALIZATION, LEWIS' DOCTORED CONDITIONAL, OR ANOTHER NOTE ON "A CONNEXIVE CONDITIONAL"

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ABSTRACT: Günther recently suggested a 'new' conditional. This conditional is not new, as already remarked by Wansing and Omori. It is just David Lewis' forgotten alternative 'doctored' conditional and part of a larger class termed neutral conditionals. In this paper, I answer some questions raised by Wansing and Omori, concerning the motivation, the logic, the connexive flavor and contra-classicality of such neutralized conditionals. The main message being: Neutralizing a vacuist conditional avoids (some) paradoxes of strict implication, changes the logic essentially only by Aristotle's Thesis, makes strong connexivity impossible, and remains in the realm of non-contra-classical logics.

KEYWORDS: neutral conditional, paradoxes of strict implication, paradoxes of material implication, definable conditional, vacuism, connexivity, super-strict Implication, contra-classicality

Wansing and Omori (2022) recently provided some historic and logical context to a proposal by Günther (2022) to define a 'new' conditional. The purpose of this note is to add more context and address some of their questions.

Günther proposes to define a conditional $A \mathbin{\Box\!\!\Rightarrow} B$ by augmenting a Lewisean conditional $A \mathbin{\Box\!\!\rightarrow} B$ by the possibility of the antecedent. Semantically, the proposal amounts to saying that $A \mathbin{\Box\!\!\Rightarrow} B$ is true at world w iff the most similar A-worlds are B-worlds *and* there is a most similar A-world. As Wansing and Omori remark, and Günther partly acknowledges, this proposal is not new.

Wansing and Omori trace the account back to Priest (1999, 145). An earlier proposal was made by Burks (1955) (cf. Pizzi 1977, 289-90). In these accounts, the underlying conditional is not a Lewisean conditional but a strict conditional. Following Gherardi and Orlandelli (2021, 2022), I call the resulting conditional (weak) super-strict implication and denote it by $\Rightarrow$.[2] The semantic definition here

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[2] Priest also suggested the stronger alternative to add the possibility of the negated consequent.

amounts to saying that $A \Rightarrow B$ is true at world w iff all accessible A-worlds are B-worlds and there is an accessible A-world. From this perspective, "it seems that Günther simply repeats for the Lewis-Stalnaker conditional what Priest suggested for a strict conditional" (Wansing & Omori 2022, 327). But Lewis (1973a, 24-6) himself already suggested to consider $A \;\Box\!\!\Rightarrow C$ as an alternative to his counterfactual $A \;\Box\!\!\rightarrow C$, more than two decades prior to Priest. He called it 'doctored counterfactual' (Lewis 1973b, 438). Thus Günther really studies Lewis' forgotten alternative doctored conditional.[3] The same idea was investigated in the related possibilistic and ranking semantics (Benferhat, Dubois, & Prade 1997; Dubois & Prade 1994; Huber 2014; Raidl 2019). Furthermore, the underlying construction is quite general: Add the assumption that the antecedent is possible to your preferred conditional. I will call the result neutralized conditional.

Such a general approach was conducted by Raidl (2020). Slightly modifying my previous terminology, let us call *neutralized conditional* $\rightarrow$ any conditional definable from a basic conditional > in the following way

$$A \rightarrow B := (A > B) \wedge \Diamond A,$$

where $\Diamond A := \neg(A > \bot)$ is the so-called outer possibility of >.[4] This is a more general syntactic definition, englobing all previous proposals. The basic conditional > is arbitrary. It need neither be a strict conditional nor a Lewisean conditional, it can be, more generally, some kind of variably strict conditional (as studied by Raidl) or a relevance conditional (as imagined by Priest).

The semantics of a neutralized conditional is as follows: $A \rightarrow B$ is true (or accepted) at world w iff the defining clause for $A > B$ holds at w and the defining clause for $\neg(A > \bot)$ holds at w. The semantics for $\rightarrow$ is only fixed, once the semantics for > is fixed. In a very weak neighborhood (sentence) selection semantics, the defining clause becomes: B is in the A-neighborhood and $\bot$ is not in that neighborhood. A belief reformulation, where the A-neighborhood is interpreted as the set of sentences believed given A, would be: B is believed given A, but $\bot$ is not. If we add some further constraints on neighborhoods or conditional beliefs, a closeness reformulation becomes available: closest A-worlds are B-worlds, and there are closest A-worlds. If closeness is analyzed in a Lewisean sphere semantics, we obtain Lewis' alternative doctored conditional (as studied by Günther). Possibilistic and ranking theoretic versions can be embedded into such

This was called s*trong super-strict implication* by Gherardi and Orlandelli, and *implicative conditional* by Gomes (2020), and Raidl and Gomes (2023).

[3] Although Günther does not fix the semantics, he speaks in terms of Lewisean similarity.

[4] Günther considers the alternative $\Diamond' A := \neg(A > \neg A)$. In his 'semantics,' the two are equivalent.

semantics, and if we suppose that there is only one sphere around each world, we obtain a semantics for (reflexive) normal weak super-strict implication. If additionally, the unique sphere is the same for each world, we obtain Priest's (S5-based) proposal. Thus all mentioned proposals are neutralized conditionals. Their underlying conditionals are just of different type or strength.

The main point in Günther (2022), however, is that neutralization is a natural way to 'connexivize' the original conditional. A similar point was made by Priest (1999, §2.5-6). However, Günther's conditional is not connexive, as Wansing and Omori remark, neither is Priest's conditional, nor any neutralized conditional, as I will show. Neutralized conditionals are rather motivated by nullifying vacuism. Instead of making an impossible antecedent conditional vacuously true, as vacuism, the neutralization makes it false. The connexive flavor is a side-effect.

The following sections echo some of the questions raised by Wansing and Omori, and provide some answers. Section 1 motivates neutralization. Section 2 presents logics for neutralizations, in particular for the neutralized weakly centered Lewisean conditional. Section 3 compares the latter to super-strict implication. Section 4 proves that connexivity is impossible for neutralizations, and Section 5 discusses contra-classicality. Non-obvious proofs are collected in the Appendix A.

1. Motivation

What is the motivation behind strengthening a conditional by the possibility of the antecedent?

Günther argues that conditionals with a contradictory antecedent are 'unintelligible' (2022, 58). Wansing and Omori rightly contest. We can very well utter and understand

(1) If it snows and it does not snow, I am the queen of England.

(2) If it snows and it does not snow, it snows.

We also reason from a contradiction without complaining about the unintelligibility of that contradiction. The problem of contradictory antecedent conditionals, and more generally, impossible antecedent conditionals, is not so much that we do not use them or that we do not understand them or their antecedents, but that our intuitions with respect to their truth or falsity, as with respect to their logical behavior are less clear than for possible antecedent conditionals.

Consider the following conditionals

(3) If 1 + 1 = 3, I'm the queen of England.

(4) If $1 + 1 = 3$, $1 + 1 + 1 = 4$.

According to a relevance-based view, (1) and (3) should be false, since there is no connection between the antecedent and the consequent. But (2) is relevantly judged true. And maybe (4) should be judged true as well. After all, if $1 + 1 = 3$ and $3 + 1 = 4$, then $1 + 1 + 1 = 4$, by adding +1 to each side, so that the (wrong) antecedent equality seems to be relevant to the (equally wrong) consequent equality.

Another view is that impossible antecedent conditionals carry another message than their cousins with possible antecedents. The meaning conveyed by (3) is not that normally or relevantly 1+1=3 implies that I am the queen of England. Besides mockery, such a conditional rather states that 1+1=3 is impossible. Let's call this the reductive view. If this were the only meaning, impossible antecedent conditionals like (3) could (and maybe should) be rephrased as simple modal statements, without loss of meaning. But some content seems lost when we rephrase any of the above (1)–(4) by '1+1=3 is impossible', as the relevance's analysis suggests. The consequent contributes to the meaning. But how? Maybe the conditional has an additional performative meaning. The conditional (rather than the modal) statement is used to illustrate the antecedent impossibility by another, often more intuitive impossibility in the consequent. Combining the reductive with the performative reading we obtain that an impossible antecedent conditional expresses the impossibility of the antecedent by illustrating it with another often more intuitive impossibility in the consequent. According to this view, it is (3) which is true (or acceptable), and rather (4) which should be false (or rejected), since in the latter, the consequent impossibility is not more intuitive than the antecedent impossibility. (Similarly (1) is true and (2) is false.)

The above are only two views for impossible antecedent conditionals. The point to present them side-by-side was merely to show that they diverge in their truth evaluation of (3) and (4). Whereas the relevance view judges the first as false and the second as true, the reductive-performative view makes the opposite judgment.

The deviance of impossible antecedent conditionals also concerns their inference behavior. For possible antecedent conditionals, many conditional accounts usually accept the following two laws:

ID	$A > A$	Identity
RW	If $\vdash B \supset C$ then $\vdash (A > B) \supset (A > C)$	Right Weakening

That is, possible antecedents imply themselves and are closed under logical implication. But it is unclear whether these laws transfer to impossible antecedent conditionals. According to relevance, ID holds but RW needs to be drastically restricted. From the reductive-performative perspective, it is ID which fails, but maybe parts of RW can be retained.

We may agree that the meaning and reasoning behavior of impossible antecedent conditionals deviates from their cousins with possible antecedents. But we may disagree on what this deviance is and how to formalize it. There are different options. We might want to judge all impossible antecedent conditionals as true – a position called *vacuism* (Williamson 2007). Conversely, we might want to judge them all as false – called *neutralism* (Raidl 2019, 2020). Hybrid options fall in between: we could suspend judgment and attribute a third truth value (for 'indeterminate'), or we might want to discriminate between some true and some false impossible antecedent conditionals (as in impossible world semantics or in relevance logic). Suitable restrictions of ID and RW will be correlated with such semantic choices. Impossible world semantics, vacuism and relevance logic all agree that impossible and possible antecedent conditionals can be treated in *the same* semantics. But they disagree whether they can be treated in the same way. Impossible world semantics treats impossible antecedent conditionals in a radically different way than possible antecedent conditionals – the former follow almost no law at all (apart from ID). Vacuism and relevance logic, on the other hand, treat both kinds in exactly the same way, the laws in vacuism being inspired by possible antecedent conditionals, whereas the laws in relevance logic are rather inspired by impossible antecedent conditionals. By contrast, I take neutralism to be a proposal for possible antecedent conditionals only, which is either in wait of completion by a suitable extension to impossible antecedent conditionals (if one thinks that the two kinds interact), or which needs to be considered as strictly separated from a theory for the latter (if one thinks that the two kinds don't interact).

Priest (1999) argued for neutralization by the 'cancellation view' of negation. Affirming a sentence and then its negation cancels both affirmations. That is, a sentence joined with its negation ($A \land \neg A$) should not entail everything, as in vacuism, nor should it entail something (A and $\neg A$), as in relevance logic, but it should entail nothing. But this restricted 'null view' only motivates neutralism half way. What about other contradictions, and impossibilities? We extend the null view from conjunctive contradictions to classical contradictions if we endorse a form of Left Logical Equivalence. The possibilistic framework based a form of neutralization on this more general null view: classical contradictions should entail nothing.[5] But neutralization rests on a much stronger claim which is just neutralism: *Impossible antecedents entail no consequent*. Lewis (1973a, 25) motivated neutralization from neutralism and although adopting vacuism, admitted that he had no decisive argument for choosing the latter.[6] A similar motivation, based on doxastic considerations, can be found in Raidl (2019).

Neutralism stands in contrast to *Vacuism*. Vacuism treats all impossible antecedent conditionals as true. For a conditional to be vacuist it suffices that it validates ID and RW (and that $\supset$ behaves classically). Let's call such a conditional *pure*. Thus pure conditionals are vacuist. But the reverse need not hold, since similar results can be proven for slightly weaker conditionals, for example where $>$ validates ID and the following deductive version of RW

dRW	If $B \vdash C$ then $A > B \vdash A > C$	deductive Right Weakening

Most conditionals are pure and hence vacuist, including the material and strict conditional, Lewisean-Stalnaker conditionals and many much weaker variably strict conditionals. Other conditionals are almost pure in that they validate ID and restrict RW (or dRW). Relevance conditionals are almost pure in this sense.

The problem with vacuist and pure conditionals is that they inherit two central paradoxes from strict implication:

AA	$\bot > C$	Antilogical Antecedent
IA	$\neg\Diamond A \supset (A > C)$	Impossible Antecedent

Almost pure conditionals may validate restricted versions of these.

The neutralization of a pure conditional avoids these paradoxes: it invalidates AA since it validates the negation NAA, and it invalidates IA, since it invalidates the inner scope negation NIA:

[5] The view is presented by the authors as if it applied to all impossibilities. But in their language, only boolean impossibilities are considered, that is classical contradictions. This is due to the fact that the authors interpret impossibility as having possibility measure 0, where the impossibility measure ranges over a boolean algebra and where additionally only (boolean) contradictions receive possibility 0.

[6] Lewis (1973b, §9) also highlighted that the doctored conditional is better suited than its vacuist cousin for analyzing conditional obligation (*Given A, it ought C*), temporal conditionals (*When next A, it will C*; *When last A, it was C*), Prior's egocentric relation (*The A is C*).

NAA	$\neg(\bot \to C)$	No Antilogical Antecedent
NIA	$\neg\Diamond A \supset \neg(A \to C)$	No Impossible Antecedent

where now the possibility needs to be expressed by $\Diamond A := (A \to \top)$.

Thus neutralization neutralizes the paradoxes of vacuist conditionals. However, since NIA entails NAA if the modality is normal,[7] the core axiom here is NIA. Yet NIA is nothing else than an object language expression of neutralism: *impossible antecedent conditionals are false*. And thus, the avoidance of the paradox IA by endorsing NIA is tantamount to adopting neutralism. In this sense, neutralization is the minimal and maybe most natural way to adopt neutralism and avoid the mentioned paradoxes of material and strict implication.

2. The Logic

> It remains to be seen, what are the particular implications when we combine the Lewis-Stalnaker conditional with Priest's framework? (Wansing & Omori 2022, 327)

The logical side of this question has been partly answered. Indeed, Raidl (2020) provided a detailed analysis, completeness results included, of neutralized conditionals in various semantics, starting from a very weak neighborhood set-selection semantics all the way up to a Lewisean (non-centered) semantics. Extending the results of that paper, we obtain that

Theorem 1. The following logic, NW, is sound and complete for the neutralized conditional in weakly centered Lewisean models:[8]

MP	If $\Gamma \vdash A$ and $\Gamma \vdash A \supset B$ then $\Gamma \vdash B$	Modus Ponens
LLE	If $\vdash A \equiv B$ then $\vdash (A \to C) \supset (B \to C)$	Left Logical Equivalence
RW	If $\vdash A \supset B$ then $\vdash (C \to A) \supset (C \to B)$	Right Weakening

PT Substitutions of classical tautologies

[7] It suffices that $\neg\Diamond\bot$ is valid.

[8] For a strongly centered semantics, we need to add the debatable law of Conjunctive Sufficiency (CS). If we want to drop $\supset$ from the language, we need to replace MP by the rules for $\wedge$ and $\neg$, and restate any axiom $X \supset Y$ in rule form $X \vdash Y$, and the rules LLE, RW in deductive form (e.g. RW becomes dRW).

AND	$(A \to B) \wedge (A \to C) \supset (A \to B \wedge C)$	Consequent Conjunction
◇ID	$\Diamond A \supset (A \to A)$	Possible Identity
AT	$\neg(A \to \neg A)$	Aristotle's Thesis
OR	$(A \to C) \wedge (B \to C) \supset (A \vee B \to C)$	Antecedent Disjunction
IOR	$(A \to C) \wedge \neg\Diamond B \supset (A \vee B \to C)$	Impossible Disjunct
RM	$(A \to C) \wedge \neg(A \to \neg B) \supset (A \wedge B \to C)$	Rational Monotonicity
TID	$\top \to \top$	Tautological Identity
MI	$(A \to C) \supset (A \supset C)$	Material Implication

In this logic, one can further derive:

wBT	$(A \to B) \supset \neg(A \to \neg B)$	weak Boethian Thesis
NAA	$\neg(\bot \to C)$	No Antilogical Antecedent
NAC	$\neg(A \to \bot)$	No Antilogical Consequent
PA	$(A \to B) \supset \Diamond A$	Possible Antecedent
N	If $\vdash A$ then $\vdash \Box A$	Necessitation
CM	$(A \to C) \wedge (A \to B) \supset (A \wedge B \to C)$	Cautious Monotonicity

The law wBT follows from AND, RW and AT. NAA follows from RW, ◇ID and AT. NAC follows from RW and AT. PA follows from RW. N follows from AT and LLE. CM follows from RM and wBT.

Note that the above neutralized conditional is really Lewis' alternative conditional $\Box\!\Rightarrow$ in a weakly centered semantics. And as long as we interpret Günther's intuitive talk of similarity in the Lewisean sense, the above is a logic for the Lewisan doctored conditional considered by Günther. To carve out the difference between $\Box\!\Rightarrow$ and $\Box\!\!\to$, note that Lewis' weakly centered conditional can be axiomatized by replacing ◇ID + TID by ID, removing AT [and IOR], but adding CM. AT is invalid for $\Box\!\!\to$, whereas ID is invalid for $\Box\!\Rightarrow$. Thus the neutralization differs from the original Lewisean conditional in that identity is restricted to tautological and possible antecedents, AT holds, CM is not required, and OR needs the additional help of IOR to make the logic complete.

By the same method, we can analyze neutralizations of weaker conditionals. For example, let's say that $>$ is an *orthodox conditional* if it is ID normal, that is, it validates ID together with the first five principles (MP)–(AND) above.[9] As corollary to Theorems 6 and 7 from Raidl (2020), we obtain:

[9] A normal conditional has a *normal conditional logic* in the sense of Chellas (1975), i.e. (MP)–(AND) together with $A > \top$, which in the presence of ID becomes redundant due to RW.

Theorem 2. The complete logic of the neutralization of an orthodox $>$ is given by the first 7 principles (MP)–(AT). And (wBT)–(N) remain derivable.

Thus the neutralization differs from the underlying conditional only in adopting AT and restricting ID. In this context, we can equivalently replace AT by wBT or by NAC.[10] And thus AT and wBT are equally at the heart of neutralizing vacuous conditionals. Further strengthenings of the logic for $>$ result in corresponding strengthenings of the logic for $\rightarrow$. For example, adding OR for $>$ results in adding OR+IOR for $\rightarrow$, adding RM for $>$ results in adding RM for $\rightarrow$, adding $\neg(\top > \bot)$ for $>$ results in adding TID for $\rightarrow$, and adding MI for $>$ results in adding MI for $\rightarrow$. The weakest neutralized logic, E, analyzed by Raidl (2020, p. 148) is given by the first four principles (MP)–(PT) together with NAC. It is the neutralized companion of the (non-normal conditional) logic given by the first four principles together with $A > \top$.

3. Comparing Neutralizations

There might be something revealing in working with a Lewis-Stalnaker conditional instead of a strict one, but that is at least not made clear in (Günther 2022). (Wansing & Omori 2022, 327)

What is the difference between neutralizing a strict conditional or a variably strict conditional? To simplify, consider a strict conditional in reflexive normal models (with the modal logic KT). How does its neutralization (the super-strict implication) differ from the neutralization of the previous Lewisean conditional? An axiomatization of super-strict implication with proof of completeness is presented by Gerhardi, Orlandelli and Raidl (2022).[11] They use the inner modality $\boxdot A := (\top \rightarrow A)$. An alternative axiomatization consists in simply augmenting the logic from Theorem 1 by the single axiom

IO	$\boxdot A \supset \Box A$	Inner to Outer modality

Theorem 3. The logic NW (from Theorem 1) augmented by IO is sound and complete for the super-strict conditional in reflexive Kripke models.

[10] AT implies NAC by RW. NAC implies wBT by AND. And wBT implies AT by RW and $\Diamond$ID. Raidl (2020) chose NAC to formalize his neutral conditional logics.

[11] These authors also axiomatize neutralizations of some non-normal strict implications.

IO is invalid for the Lewisean neutralization, but the reverse Outer to Inner modality (OI) is valid. Thus both neutralizations just differ by a single axiom.[12]

There are further differences. For example, super-strict implication validates a version of Transitivity, and restricted versions of Contraposition and Strengthening the Antecedent:

wTR	$(A \to B) \wedge (B \to C) \supset (A \to C)$	weak Transitivity
PC	$\Diamond\neg B \wedge (A \to B) \supset (\neg B \to \neg A)$	Possibilistic Contraposition
PM	$\Diamond(A \wedge B) \wedge (A \to C) \supset (A \wedge B \to C)$	Possibilistic Monotonicity

These are invalid for the neutralized Lewisean conditional.[13] Simply by construction, super-strict implication is 'closer' to strict implication than the neutralized Lewisean conditional, which in turn is closer to its underlying conditional.

4. Impossible Connexivity

> Günther's conditional is *not* connexive. It does, however, have some connexive flavour" (Wansing & Omori 2022, 325)

A conditional is called *connexive*,[14] if it invalidates Symmetry

S $(A \to B) \to (B \to A)$,

and validates AT and

BT $(A \to B) \to \neg(A \to \neg B)$. Boethius Thesis

It is called *Kapsner strong* if the following hold

Unsat1. In no model is $A \to \neg A$ satisfiable,

Unsat2. In no model are $A \to B$ and $A \to \neg B$ satisfiable.

It is *strongly connexive* if it is connexive and Kapsner strong. If negation and $\supset$ are classical, then Unsat1 and Unsat2 are respectively equivalent to AT and wBT. Let's

[12] This difference really boils down to the underlying conditionals – strict or Lewisean. The inner and outer modality of a Lewisean conditional are distinct: $\boxdot A = (\top > A)$ and $\Box A = (\neg A > \bot)$. These are equivalent for the strict conditional. But otherwise, the latter validates the same principles as a weakly-centered Lewisean conditional.

[13] An essential difference between (weak) super-strict implication and strong super-strict implication, is that the latter validates Aristotle's second Thesis (AT2) $(A \to B) \supset \neg(\neg A \to B)$, which is invalid for (weak) super-strict implication. For an axiomatization of strong super-strict implication in reflexive Kripke models, see (Raidl & Gomes 2023).

[14] McCall (1963, 1966) and Wansing (2022).

call a conditional *pseudo-connexive* if it invalidates S and validates AT and wBT. It is *strongly pseudo-connexive* if additionally it is Kapsner strong.

Günther's (Lewis' doctored) conditional is not connexive, since it invalidates Boethius' thesis, as noted by Wansing and Omori. However, it is pseudo-connexive and due to classicality of $\neg$ and $\supset$ it is strongly pseudo-connexive.[15]

This will hold for many neutralizations of conditionals with a consistent logic. For Unsat2 it suffices that the underlying conditional $>$ validates the deductive version dAND of AND (built in a similar way from AND as dRW from RW) and dRW applied to $B \wedge \neg B \vdash \bot$. For Unsat1, it suffices that $>$ additionally validates ID.[16] For AT it then suffices that additionally $\neg$ is classical, and for wBT it suffices that $\supset$ is also classical. For invalidity of S it suffices that the underlying $>$ validates ID and dRW applied again to $B \wedge \neg B \vdash \bot$.[17] Let's say that $>$ is *conjunctive*, if it validates ID, dAND, and dRW applied to $B \wedge \neg B \vdash \bot$.

Then we obviously have:

Theorem 4. Let $\rightarrow$ be the neutralization of $>$.

- If $>$ is conjunctive, then $\rightarrow$ is Kapsner-strong and invalidates S.
- If additionally $\neg$, $\supset$ are classical, then $\rightarrow$ is (strongly) pseudo-connexive.

From this perspective, the distinction between pseudo-connexivity and strong pseudo-connexivity (by adding 'Kapsner strong') does not make much sense, since as soon as pseudo-connexivity is ensured by classicality of $\neg$ and $\supset$, the conditional is automatically Kapsner strong. Thus, from the perspective of neutralizations, one rather approximates connexivity by the following steps: first ensure Unsat2 (by dAND and dRW for $>$), then Unsat1 (by ID for $>$), and thereby invalidity of S. Classicality of $\neg$, $\supset$ then ensures AT and wBT. Hence rather than being a strengthening of pseudo-connexivitiy, being 'Kapsner strong' is a precondition of pseudo-connexivity.

From the above result, it follows that the 'connexive flavor' of neutralizations of orthodox conditionals is that they are strongly pseudo-connexive. One might think that we then only have one step to go to obtain a connexive conditional: add Boethius' thesis. However this is impossible:

Theorem 5. Adding BT to a pure neutralized conditional logic is inconsistent.

[15] That $\Rightarrow$ validates AT, the deductive version of wBT, and some other principles was noted by Priest (1999).

[16] If one takes the alternative outer modality, Unsat1 follows by definition, but Unsat2 requires dRW additionally.

[17] The special case $((\bot > \bot) \wedge \neg(\bot > \bot)) > ((\bot > \bot) \wedge \neg(\bot > \bot))$ of ID suffices.

Proof. A pure conditional is given by MP, PT, RW, ID. The neutralization of a pure conditional still validates AT, PA, and N. BT implies $\Diamond (A \to C)$ for any A, C, by PA. Thus $\Diamond (\top \to \bot)$. But $\neg(\top \to \bot)$ by AT. Hence $\Box\neg(\top \to \bot)$ by N. That is $\neg\Diamond(\top \to \bot)$. QED.

It's not just that neutralization does not give us new insights into connexivity, connexivity is incompatible with neutralization. BT is not only invalid, but strongly invalid, since any BT extension of a pure neutralized conditional logic is inconsistent. For the same reason, neutralized conditionals will (strongly) invalidate any nested law of the form $(A \to B) \to C$. The strong invalidity of S and BT fall into the same basket. The problem concerns a vast class of neutralized conditionals. Only neutralizations of impure conditionals (non-ID or non-RW) escape. But impure conditionals don't create the vacuist problems (AA, IA) for the avoidance of which neutralization was conceived in the first place! The only comfort we may take in neutralized conditionals (apart from being pseudo-connexive), is maybe that they validate the outer-scope version of BT

oBT. $\neg((A \to B) \to (A \to \neg B))$ outer scope Boethian Thesis

For this $\Diamond$ID and wBT [i.e. AT, AND, RW] suffice.

The more intricate worry about connexivity is as follows. The combination of the standard principles RW and ID is incompatible with AT and also with wBT. Indeed, if ID would hold, $\bot \to \bot$ would hold and by RW $\bot \to \top$ would hold. But this contradicts AT (it also contradicts wBT). Thus upholding ID and RW together is not compatible with AT (nor with wBT). Hence either ID or RW need to go, for a connexive conditional. Neutralization restricts ID but keeps RW, the result being that it makes connexivization impossible (Theorem 5). Thus, maybe if we have learned something it is that neutralization will not help in the study of connexive logic, and that ultimately, we should better explore the route where we keep ID but drop or restrict RW. This is basically the relevantist route.

5. Contra-Classicality?

If neutralization does not lead to (strong) connexivity, then at least, it may be one way of exploring contra-classical logics, as Wansing and Omori suggest.

> [...] a simple variant of Lewis conditional will bring us to the realm of contra-classical logics (cf. (Humberstone 2000)). The same applies to the variants of strict implications explored by Gherardi and Orlandelli, and this seems to be a simple and interesting route to contra-classicality. (Wansing & Omori 2022, 326-7)

I will argue that this is only true in a very restricted sense, and that contra-classicality is not the appropriate notion to characterize logics of neutralizations (or related constructions).

In general, a neutralized logic, say NL, arises from a companion conditional logic L for some underlying conditional >. The neutralized logics considered here are extensions of classical propositional logic CL (since L extends classical logic), thus they are *not* contra-classical in the sense of being incompatible with classical logic. They verify if $\vdash_{\mathsf{CL}} \alpha$ then $\vdash_{\mathsf{NL}} \alpha$ and also the converse for α a classical sentence. However, the neutralized logics are contra-classical in another, very strict sense: Call *t the literal translation* if the conditional → is translated into the material conditional ⊃ and *t* preserves Booleans and propositional variables. A propositional logic S with a new conditional-like connective → is *literally contra-classical* iff the literal translation *t* does not satisfy

$$\text{If } \vdash_{\mathsf{S}} \alpha \text{ then } \vdash_{\mathsf{CL}} t(\alpha) \tag{5.1}$$

The neutralized logics are literally contra-classical, since AT (or wBT) is literally translation resistant, i.e., it is derivable in the neutralized logic, but classically invalid under the literal translation, and thus not classically derivable. Thus → cannot receive the classical material conditional interpretation. But literal contra-classicality is not the notion Wansing and Omori had in mind.

A propositional logic is *contra-classical* iff it is not a sublogic of classical propositional logic, not even modulo a translation which preserves propositional variables. Yet contra-classicality without some restriction (called 'profound') is too restrictive since it reduces to the notion of inconsistency (Humberstone 2000, Proposition 1.1). But we can require the translation to preserve Booleans (¬, ∧, ∨, ⊃, ⊤, ⊥), and speak of contra-classicality *modulo Booleans*. Literal contra-classicality is a special case, and Humbersone's notion of contra-classicality (modulo Booleans) simply extends literal contra-classicality by testing (5.1) for other translations than the literal one. What is really being tested thereby is whether → can receive any classical interpretation at all. But the neutralized logics are not contra-classical in this sense either, as we will now see.

Neutralizations are definable conditional constructions from some basic conditional > (Raidl 2020, 2021). This is to say that there is a translation o from the language of → to the language of >, preserving Booleans and propositional variables and such that scheme (5.1) holds from NL to L, modulo o. The translation of neutralizations arises naturally by using the semantic definition. It is induced from

$$(A \rightarrow B)^{\circ} := (A^{\circ} > B^{\circ}) \wedge \neg(A^{\circ} > \bot)$$

meaning that all standard connectives are normally translated, and propositional variables remain untranslated. Thus the translation preserves Booleans. Furthermore, one can prove that if $\vdash_{\mathsf{NL}} \alpha$ then $\vdash_{\mathsf{L}} \alpha^{o}$. In the above terminology: NL is not contra-L modulo Booleans.

Whether NL is contra-classical modulo Booleans reduces to the question of whether L is contra-classical modulo Booleans. But neither the Lewisean weakly centered logic (VW), nor the logic of normal strict implication are contra-classical modulo Booleans. We can indeed translate the Lewisean > into ⊃ – denote the translation $\#$ – and satisfy (5.1) for S = VW and $t = \#$.

That is, > can be interpreted classically and in fact literally (although this is not the intended interpretation). (Similarly for a normal strict implication.) Chaining o and #, we then obtain that → translates into ∧ and $t = o\#$ still respects (5.1) for S = NL. Hence → can also be interpreted classically, but not literally, and the ∧-interpretation is of course not the intended one.[18] Hence the neutralized logics are *not* contra-classical (modulo Booleans), either. A similar remark holds in general for other conditional constructions out of normal conditionals.[19]

Overall, neutralization does not generate contra-classical logics out of logics which are not contra-classical. Contra-classicality of the conditional construction may at best be inherited from the underlying conditional, not from the construction. If at all, neutralization allows to construct new contra-classical logics from already existing contra-classical logics.

An example is the neutralization of an S6 strict implication. The modal logic S6 can be seen as S2 augmented by the axiom $\neg\Box\Box A$. Gherardi, Orlandelli, and Raidl (2022) present a complete axiomatization (ST2) of the neutralization of S2 strict implication. The neutralization of S6 strict implication only requires to add the axiom $\neg\boxdot\boxdot A$ (that is $\neg(\top\to(\top\to A)))$ to ST2. Since S6 is a consistent contra-classical modal logic (Humberstone 2000, Proposition 2.1), the neutralization is also consistent and contra-classical. The reason here is the backtranslation • of □ into super-strict implication, induced by $(\Box A)^\bullet = \boxdot A^\bullet$. We have: if $\vdash_{\mathsf{S6}} A$ then $\vdash_{\mathsf{ST6}} A^\bullet$, analogously to Lemma 2 of Gherardi, Orlandelli, and Raidl (2022) for S2 and ST2. Thus if ST6 were not contra-classical, then we would have a translation T, such that $\vdash_{\mathsf{ST6}} B$ implies $\vdash_{\mathsf{CL}} T(B)$, and hence a translation $t' = \bullet T$, such that (5.1) holds for $t = t'$ and S = S6. But then S6 would not be contra-classical, contrary to

[18] The ∧-interpretation can however be used to show that NL is consistent (has a model), and to find non-derivable formulas.

[19] This is analogous to Humberstone's remark that there are no consistent normal modal logics which are contra-classical modulo Booleans.

Humberstone's result. Contra-classicality is here due to the non-congruentiality, which is transferred from S6 to its neutralization. In short, the neutralization of an S6 strict conditional has no classical truth-functional interpretation whatsoever.

Finally, if we slightly stretch the notion of classicality and count first order logic as classical, then we lose contra-classicality altogether. As long as the underlying conditional is first order translatable, the neutralized conditional is as well. One then obtains that if $\vdash_{\mathsf{NL}} \alpha$ then $\Gamma \vdash_{\mathsf{FOL}} \forall x\, \alpha^*$, under suitable assumptions Γ on the relations used for the first-order translation.[20] In particular, since the Lewisean conditional and strict implication are first order translatable, the conditional construct is also first order translatable. Thus these conditional constructions are not contra-classical in the first order sense either.

For these reasons, I see definable conditional constructions rather as a way to explore semantic strengthenings (or weakenings) or mixtures of existing conditionals. The conditional construction comes immediately with a proper axiom for the definable construction. For the neutralized conditional, the proper axiom is AT, or wBT, or NAC (depending on how one sees it). In view of this and Theorem 4, neutralization is essentially pseudo-connexivization, but nothing more on the connexive hierarchy, by Theorem 5.

A. Proofs

Proof of Theorem 1. Raidl (2020, Corollary 1) proved that the logic, say NV, given by MP, PT, LLE, RW, AND, NAC, $\Diamond$ID, $\Box$M, OR, IOR, RM is sound and complete for the neutralized conditional in Lewisean models (where $\Box$M is the monotonicity axiom $\Diamond A \supset \Diamond(A \vee B)$). By the same proof procedure, we can obtain a complete logic for weakly centered Lewisean models. For this it suffices to recall that (1) the weakly centered Lewisean conditional has the logic VW and extends the logic V of the Lewisean conditional by the axiom MI, and that (2) the backtranslate of MI is of the form $((A \to B) \vee \neg(A \to \top)) \supset (A \supset B)$ and can be decomposed into MI and $\neg(A \to \top) \supset \neg A$, the contraposed of which is $A \supset (A \to \top)$. From this TID follows. Conversely TID and MI together with the remaining axioms imply $A \supset (A \to \top)$: Assume A. Thus $\neg(\top \supset \neg A)$. Hence $\neg(\top \to \neg A)$ by MI. But $\top \to \top$ by TID. Thus $A \to$

[20] For the first order translation of a KT strict conditional we need to assume that R is reflexive. For a first order translation of a (weakly centered) Lewisean conditional, we need to encode the semantic assumptions on the accessibility relation R and the similarity relation ($R'xyz$ iff $y \precsim_x z$) in first order language – the binary relation R is reflexive, and the ternary R' when restricted to its first component $R'x$ is a total preorder over R-accessible points from x, such that Rwv implies $R'wwv$. All these constraints are first order definable.

$\top$ by RM and LLE. Hence NV+MI+TID is sound and complete for $\rightarrow$ in weakly centered Lewisean models.
It now suffices to show that NV+MI+TID is equivalent to our NW.

First we show that we can derive PA and AT from NV+MI+TID.

PA. Suppose $A \rightarrow B$. Hence $A \rightarrow \top$ by RW. This is $\Diamond A$.

AT. Suppose $A \rightarrow \neg A$. Then $\Diamond A$ by PA. Thus $A \rightarrow A$ by $\Diamond$ID. Hence $A \rightarrow \bot$ by AND. This contradicts NAC. Therefore $\neg(A \rightarrow \neg A)$.

Second let us conversely show that our NW derives NAC and $\Box$M.

NAC. Suppose $A \rightarrow \bot$. Hence $A \rightarrow \neg A$ by RW. This contradicts AT. Hence $\neg(A \rightarrow \bot)$.

$\Box$M. Suppose $A \rightarrow \top$. If $\neg(B \rightarrow \top)$, that is $\neg\Diamond B$, then $A \vee B \rightarrow \top$ by IOR. If on the other hand $B \rightarrow \top$, then $A \vee B \rightarrow \top$ by OR. QED.

Proof of Theorem 3. Gherardi, Orlandelli, and Raidl (2022, Theorem 18) proved that the following logic, SST, for super-strict implication is sound and complete in reflexive Kripke models: MP, PT, LLE, RW, AT, $\diamond$PA, INC, AND, TID, SPRES, $\boxdot$T, where

$(A \rightarrow B) \supset \diamond A$	$\diamond$PA
$(A \rightarrow B) \supset \boxdot(A \supset B)$	INC
$\boxdot(A \supset B) \wedge \diamond A \supset (A \rightarrow B)$	SPRES
$\boxdot A \supset A$	$\boxdot$T

We show that SST is equivalent to NW+IO (i.e. replacing $\diamond$PA, INC, SPRES, $\boxdot$T by $\Diamond$ID, OR, IOR, RM, MI, IO).

First we show that $\Diamond$ID, OR, IOR, RM, MI, IO are derivable in SST. $\diamond$ID, OR, RM, MI were shown derivable (Gherardi et al., 2022, Lemma 11). It remains to derive $\Diamond$ID, IOR, IO, and OI.

IO. We show the contraposed $\Diamond A \supset \diamond A$. Assume $\Diamond A$. That is $A \rightarrow \top$. Hence $\diamond A$ by $\diamond$PA.

OI. We show the contraposed $\diamond A \supset \Diamond A$. Assume $\diamond A$. That is $\neg(\top \rightarrow \neg A)$. But $\top \rightarrow \top$ by TID. Thus $A \rightarrow \top$ by RM. This is $\Diamond A$.

$\Diamond$ID. Assume $\Diamond A$. Thus $\diamond A$ by IO. Hence $A \rightarrow A$ by $\diamond$ID.

IOR. Suppose $A \rightarrow C$ and $\neg\Diamond B$. Thus $\boxdot(A \supset C)$ by INC, $\neg\diamond B$ by OI, and $\diamond A$ by $\diamond$PA. From $\neg\diamond B$ we obtain $\boxdot\neg B$ and hence $\boxdot(B \supset C)$, by standard reasoning with $\boxdot$ (a KT necessity). Thus also $\boxdot(A \vee B \supset C)$, and $\diamond(A \vee B)$, again by standard reasoning with $\boxdot$. Hence $A \vee B \rightarrow C$ by SPRES.

Second, and conversely, let us derive $\diamond$PA, INC, SPRES, $\boxdot$T from NW+IO.

$\diamond$PA. Suppose $A \to B$. Thus $\Diamond A$ by PA (i.e. RW). Hence $\diamond A$, contraposing IO.

INC. Suppose $A \to B$. Thus $A \to (A \supset B)$ by RW. If $\neg\Diamond\neg A$, then $\top \to (A \supset B)$ by IOR and LLE. If $\Diamond\neg A$, then $\neg A \to \neg A$ by $\Diamond$ID. Hence $\neg A \to (A \supset B)$ by RW. Therefore $\top \to (A \supset B)$ by OR. Thus overall $\boxdot(A \supset B)$.

$\boxdot$T. Suppose $\top \to A$. Hence $\top \supset A$ by MI. That is A.

SPRES. Suppose $\top \to (A \supset B)$ and $\neg(\top \to \neg A)$. Then $A \to (A \supset B)$ by RM. Hence $\Diamond A$ by PA. Thus $A \to A$ by $\Diamond$ID. Therefore $A \to (A \wedge B)$ by AND and RW. Hence $A \to B$ by RW again. QED.

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