REMARKS ON POINCARÉ' NOTION OF MATHEMATICAL RIGOUR

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ABSTRACT: Between 1906 and 1911, as a response to Betrand's Russell's review of *La Science et l'Hypothèse*, Henri Poincaré launched an attack on the movement to formalise the foundations of mathematics reducing it to logic. The main point is the following: the universality of logic is based on the idea that their truth is independent of any context including epistemic and cultural contexts. From the free context notion of truth and proof it follows that, given an axiomatic system, nothing new can follow. One of the main strategies of Poincaré's solution to this dilemma is based on the notions of understanding and of grasping the architecture of the propositions of mathematics. According to this view mathematic rigour does not reduce to "derive blindly" without gaps from axioms, mathematical rigour is, according to Poincaré, closely linked to the ability to grasp the architecture of mathematics and contribute to an extension of the meaning embedded in structures that constitute the architecture of mathematical propositions. The focus of my paper relates precisely to the notion of architecture and to the notion of understanding. According to my reconstruction, Poincaré's suggestions could be seen as pointing out that understanding is linked to reason not only within a structure but reasoning about the structure.

KEYWORDS: architecture, mathematical rigour, logic and mathematics

Between 1906 and 1911, as a response to Betrand's Russell's review of *La Science et l'Hypothèse*, Henri Poincaré launched an attack on the movement to formalise the foundations of mathematics reducing it to logic. The main point is the following: the universality of logic is based on the idea that their truth is independent of any context including epistemic and cultural contexts. From the free context notion of truth and proof it follows that, given an axiomatic system, nothing new can follow. If mathematics is reducible to logic, then there is no place for creation. Philosophers would express this in the following way: logical proofs are analytic, that is provide no new information beyond the the premises or axioms, but mathematics does provide information: mathematics is thus synthetic and hence different of logic.

As pointed out by Detlefsen[1], one of the main strategies of Poincaré's solution to this dilemma is based on the notions of *understanding* and of *grasping the architecture* of the propositions of mathematics. According to this view mathematic rigour does not reduce to "derive blindly" without gaps from axioms, mathematical rigour is, according to Poincaré, closely linked to the ability to grasp the architecture of mathematics and contribute to an extension of the meaning embedded in structure that constitute the architecture of mathematical propositions. The focus of my paper relates precisely to the notion of architecture and the notion of understanding. I will suggest a diachronic and synchronic reconstruction of the notion of architecture – the latter considers the architecture as a cultural object. Actually what I will try to do is to link Poincaré's arguments against the logicians with his paper *La science et les humanités* of 1911 where he argues that the development of the ability to grasp the architecture (intuition) must be studied as the result of the refined ability of understanding acquired by means of the practice of humans sciences in a given culture.

To expresses it bluntly, according to Poincaré, mathematics is intimately related to culture because it is about the construction of a structure of relations between propositions and this structure is not universally given, but developed within the cultural conventions of a community.

1. The Problem and Poincaré's solution: Rigour in Mathematics and Rigour in Logic

In a manner reminiscent of Kant's opening remarks to the First Part of the Transcendental Problem of the Prolegomena, Poincaré opens *La Science et l'Hypothèse* with these words:

> La possibilité même de la science mathématique semble une contradiction insoluble. Si cette science n'est déductive qu'en apparence, d'où lui vient cette parfaite rigueur que personne ne songe à mettre en doute? Si, au contraire, toutes les propositions qu'elle énonce peuvent se tirer les unes des autres par les règles de la logique formelle, comment la mathématique ne se réduit-elle pas à une immense tautologie? Le syllogisme ne peut rien nous apprendre d'essentiellement nouveau et, si tout devait sortir du principe d'identité, tout devrait s'y ramener. Admettra-t-on donc que les énoncés de tous ces théorèmes qui remplissent tant de volumes ne soient que des manières détournées de dire que A est A? ... Si l'on se refuse à admettre ces conséquences, il faut bien concéder que le raisonnement mathématique a par lui-même une sorte de vertu créatrice et par conséquent qu'il se distingue du syllogism.

[1] Michael Detlefsen, "Poincaré Against the Logicians," *Synthese* 90, 3 (1992): 349-378.

> La vérification diffère précisément de la véritable démonstration, parce qu'elle est purement analytique et parce quelle est purement stérile. Elle est stérile parce que la conclusion n'est que la traduction des prémisses dans un autre langage.[2]

The dilemma seems to be linked to the notion of mathematical rigour:

1) Mathematics is perfectly rigorous,

2) Mathematical proofs are not merely logical inferences. Furthermore, conclusions of mathematical proofs can, and often do, constitute extensions of the mathematical knowledge represented by the premises. Thus, mathematical proofs do not seem to be purely logical.

Hence, though mathematical proof is rigorous it is not reducible logical rigour. The point now is to specify what rigour is.

Important is to see that with this formulation we would like to avoid to reduce Poincaré's point to the trivial remark that the axioms of mathematics are indeed not logical but everything else follows logically from them. It looks that Poincaré links mathematical rigour with *mathematical understanding* or *mathematical insight* (perspicacité et pénétration), that is topic-specific knowledge. It is not by leaving some gaps in a demonstration that qualifies it as non rigorous but because of lack or mathematical insight (perspicacité et pénétration) or understanding of the mathematical object.

Moreover, Poincaré formulates this as a general epistemological problem. Poincaré idea is that given a set of mathematical axioms, the inferences of the mathematicians have a distinctive epistemological feature which distinguishes it from the inferences drawn by a logician from the very same axioms. Leaving by side the qualification of synthetic to the inferences drawn by the mathematician and of analytic of those drawn by the logician is that the notion of knowledge involved is different. The notion of knowledge involved by the mathematician is strongly linked with understanding the mathematical field while the notion of the logician is so to say contextually independent. In other words, knowledge of a body of mathematical propositions, plus mastery over their logical manipulation, does not amount to mathematical knowledge either of those propositions or of the propositions derived from them.

[2] Henri Poincaré, *La science et l'hypothèse* (Paris: Flammarion, 1902), 9-13.

It really looks as Poincaré is aiming at a much more general epistemological point that has too close links with Kant, namely, that different kind of sciences might have a proper notion of inference and because Frege's Russell's logic is based on a general notion of inference this makes it, on Poincaré's view, trivial.

But what is this mathematical understanding or insight (perspicacité et pénétration) of the mathematical object? How is this achieved? Here is Poincaré less precise and makes use of three notions that triggered important developments, namely: the notions of

- construction,
- intuition,
- system or architecture.

The leading idea here is of system. Once more a Kantian topic: each science has its own architectonic or system that consists on non logical relations between propositions. Knowledge of this architecture is knowledge to produce these relations and create new ones, here does Poincaré speak of *intuition*. A mathematical proof is related to establish a link between the architecture in which the premises are embedded and the architecture of the conclusion. Poincaré calls this type of knowledge "intuition". Different to Kant, Poincaré does not think that this architectonic is given a priori: it is a synthetic process by which the system is *constructed*. Voilà here we have the three notions mentioned above.

Certainly though challenging this is not precise enough, let me know briefly mention what Brower and the intuitionists made of these remarks.

2. The Structure of the Domain and the Intuitionistic Interpretation

Let me express the intuitionistic interpretation and further development of Poincaré's remarks beyond perhaps his own ideas in the following way: Kant's great contribution consisted in realizing not only that mathematics as every other science has its own characteristic architectonic that systematizes it but also that mathematics has a special structured domain. On this view, the domain of mathematics is being structured by time. Thus mathematical objects are constructions and rigorous inferences are those that always keep track of the construction of the objects the propositions involved are about. Brower interpreted Poincaré's appeal to *intuition of the structure of the domain* as *experience* of the mathematical object, meaning: the experience of constructing the object at stake.

In this framework, the proof by mathematical induction has central place: it is the most typical way of proving adequate to mathematical constructions. Proof by mathematical induction is precisely Poincaré's most cherished example of a rigorous mathematical inference that is not logical but purely mathematical.

Despite Brower's own sceptical attitude towards logics intuitionistic logicians, particularly Ardent Heyting, went a step forward and dared to describe a logical system that carries the structure of the domain to the structure of the propositions.

For the first time a logical system was not seen only as pure logical relation between propositions but as relation where the epistemic subject is introduced. Logical relations are not seen as being established by logical consequence, but by inference, where inference is the relation between propositions but between judgements, and judgements carry the epistemological structure of the domain. That is, the formal structure of inferences should be based on the constructions of the domain. In other words, mathematical objects are the result form constructions and this applies to proofs too. Time thus structured the domain of objects and the inferential relations between judgements. This has as consequences that some venerable logical axioms and logical proofs based on those axioms will fail, namely, third excluded, double negation and indirect proofs such as via absurdum.

Notice that the development of a logic that claims to be based on the idea of the structure of the domain seem to work against not only of Brower's but also against Poincaré's rejection of logics as describing mathematical proof.

The development of an inference system that carries in its deductive structure the epistemic structure of the system of mathematics was linked too to some remarks of Poincaré where he compares the knowledge of the winning strategy in a chess game with the knowledge of the way to construct a proof. In analogous way that it is not enough to know that there is a winning strategy to win the game, it is not enough to know that there is a proof to say that mathematical proof has been performed. We must be able to show how to construct this proof. A proof beyond our abilities to construct it is not proof at all. Intuitionistic epistemologists linked this idea to the challenge of the truth as given: a truth beyond our abilities to find it can not provide the foundations of the notion of inference. It is rather the other way: human playable or reachable proof provides the foundation to the notion of inference and truth.

Michael Dummett developed intuitionistic logic into a general conception of logic beyond mathematics and antirealism was born. Dummett and Hintikka brought into the discussion Wittgenstein's language games that provided a more precise framework to work out the notion of human playable. Indeed if language games are to work as a benchmark for the studying language and even to function as meaning mediators between language and world, these games have to be humanly playable games. Humanly playable language games were linked by the dialogical and the game theoretical tradition of Hintikka to Poincaré's and the intuitionist notion of a humanly constructible proof.

Now is that the end of Poincaré's epistemological project? Was Poincaré with the words of Brower a pre-intuitionist like Borel and Lebesgue who motivated or pre-announced the intuitionistic movement and we should go further on without him? Interesting is that Poincaré did not in fact claim against any particular logical law, but rather insisted in the notion of inference in a system and as developing the system or architecture. Let me now push this idea forward.

3. Structures and Modality

According to my reading Poincaré has a double strategy: the first strategy consists arguing from philosophy to mathematics and the second from mathematics to philosophy. From the first strategy it results that mathematics is mainly an act, a construction and from this point of view it is synthetic. From the second strategy it is an object, the result of the construction and from this point of view it is analytic and can be done in abstraction of the context where this construction was achieved.

The very point of Poincaré's argument against the logicians is that systems of sciences are not only a set of propositions related by logical consequence. There are other, extralogical or metalogical relations which build the structure of the corresponding science. The structure of these propositions might indeed be based on the structure of the objects the propositions of that science are about, like in the intuitionistic interpretation. But the idea is more general than that and I think it could be understood when related to the recent structural approaches to modal logic

3.1. Inferences within and about a structure

Let us recall that the truth definition of modal logic tell us what formulae are true in what possible world of any given model.[3] The valuation function of such model gives

[3] **DEFINITION**: Model, Frame, Truth.

A model $<W,R,v>$ for modal propositional logic consists of

a non empty set W of positions (traditionally interpretedc as possible worlds, contexts or scenarios: like temporal states, states of information etc.)

a binary relation R on W called *accessibility relation*

a valuation function v which assigns a truth value v(a) to each propositional letter of the propositional language in each position $w \in W$

us the values of the propositional letters and the truth definition extends this to the complex formulae. The difference of this truth definition to the classical case is that the truth is here made relative to the value of the positions in the structure of the model at stake. Furthermore the evaluation is dependent too on the interrelations between the given positions in that structure.

And here we are; modal logic displays the interrelation between inference and structure in such a way that each structure yields its own notion of inference. Moreover, one can at the object language level display axioms that describe the structure, usually given at the meta-language level. This is called frame validity. In this framework we would say that Poincaré is searching for those inferences the result of which describes the structure. The modern modal logician would say that Poincaré is searching for object language laws to characterize frame validity.

We should then distinguish between the logic of the model, that is, driving the logical consequences within the structure, what Poincaré might want to call the purely logical manipulation of the propositions **in** the structure (that amounts to truth in a model), and the use of propositions at the language level to describe the structure in which this propositions are embedded (truth in the frame).

3.2. The structure of propositions and the structure of the domain as an object

What happens if we would like to describe the structure of the domain? At this point we meet the famous Barcan formulae that in the philosophical tradition regulate the passage from possibility to existence and from the purely structural point of view describe one particular structure of the domain.

If the propositional frame is extended with a structure where the domains of each position at the structure are (at least) decreasing then the passage from possibility to existence is assured:

$$\Diamond(\exists x)Px \square (\exists x)\Diamond Px$$

A set W with a suitable accessibility relation is called a *frame* or *structure*. Thus given a frame <W,R> we can turn it into a model by the addition of the valuation function *v*. Moreover any given frame can be turned into a variety of different models, depending on the valuation function which is added. For a frame only establishes the positions we are dealing with and fixes which are accessible from which. A valuation is needed to establish what is the case in each of the possible positions and in general there will be many ways to do that. Each of this ways is a model establishing the factual conditions under which our logical explorations will take place. The frame will provide the basis of anyone of a variety of such factual conditions.

If the domains are not decreasing (and not constant) then the formula does not hold. Moreover if the domains are at least increasing the inverse Barcan formula holds. That is the inverse Barcan formula describes at the object language level a so to say constructive property of the domain.

$(\exists x)\Diamond Px \rightarrow \Diamond(\exists x)Px$

Certainly if both hold then there is no construction: the domain is constant!

Now, does not the latter hold too for intuitionist first order logic? The point is here that the Barcan formulae describe the structure of the domain independently of the structure of the propositions! We have a way to describe the domain without touching the classical propositional validity of any logical law.

4. Structure as an Act: Creativity

We are assuming that the structure is given, now, let us drop that assumption. The point is then the following. Let us assume that because of topic-related knowledge, including perhaps some no complete knowledge of the structure involved, we take that a given proposition is true and even valid, but we do not have a complete description of the structure. Then we could ask the following question: how should this structure be completed or how should it be if the given proposition has to be valid. This will takes us to a kind of structural abduction that is indeed not reducible to pure logical inference. Completing a conceptual structure within mathematics extends mathematical knowledge and this is what creativity is all about.

On my view the whole movement triggered by Poincaré and Brower relates to one deep epistemological point: the core result of the building of mathematics and logic were achieved by means of the creative effort of human imagination. Mathematics and logic are creation in the same sense that art is. The challenge to fully understand the epistemological implications of this point are still there and they do not seem to stop to fascinate and puzzle us again and again.